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Coulomb-Like Interactions in the Dirac Equation

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The Dirac equation for the Coulomb-like problem is modified by incorporating minimal interactions into the Dirac Hamiltonian, that keep the $1/r$ potential dependence. We determine the general energy eigenvalues and the corresponding eigenfunctions.

Let us consider a general quantum system described by canonical coordinates Q_i and P_j satisfying the Heisenberg algebra (Bruce *et al.*, 1996)

$$[Q_i, P_j] = i\hbar\mathbb{I}\delta_{ij}, \tag{1}$$

where $\mathbb{I} \equiv I_{n\times n} \otimes I$ represents an n-block identity matrix, such that we may express these operators in the general form

$$Q_i = \widehat{\eta} \otimes q_i, \qquad P_j = \widehat{\eta} \otimes p_j, \tag{2}$$

where $p_j = -i\hbar\partial/\partial q_j$. Here $\widehat{\eta}$ is a constant $n \times n$ hermitian matrix operator satisfying $\widehat{\eta}^2 = I_{n\times n}$. From Eq. (2) we can define a *label* Δ associated with each representation of the Heisenberg algebra (1)

$$n \geq \Delta(Q_i, P_j) \equiv |\mathrm{Tr}(\widehat{\eta})| \geq 0. \tag{3}$$

Representations satisfying $\Delta = n$ correspond to the usual ones ($\widehat{\eta} = I_{n\times n}$), where Q_i and P_j are reducible operators for $n \geq 2$.

The Hilbert space is defined as $\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathbb{C}^n$. It consists of n-component column vectors where each component ψ_i is a complex valued function of the *four*-dimensional (flat) space-time coordinates $\mathbf{q}$ and t. The scalar product is given by

$$(\Psi, \Phi) \equiv \int_{\mathbb{V}\subset\mathbb{R}^3} \sum_{i=1}^{n} \psi_i^*(\mathbf{q}, t)\phi_i(\mathbf{q}, t)\, d^3q. \tag{4}$$

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The operator $\mathbf{Q}$ consists of three self-adjoint operators Q_i whose domains are defined as

$$\int_{\mathbb{V}\subset\mathbb{R}^3} (Q_i\Psi)^\dagger Q_i\Psi\, d^3q = \int_{\mathbb{V}\subset\mathbb{R}^3} \sum_{j=1}^{n} |q_i\psi_j|^2\, d^3q < \infty. \tag{5}$$

The momentum operator $P_j = -i\hbar\widehat{\eta}\otimes\partial/\partial q_j$ can be defined as the Fourier transform of the position operator $Q_j (j = 1, 2, 3)$.

Minimal interactions can now be introduced by means of the prescription $P_\mu \to P_\mu - gA_\mu$, where g is the coupling constant and A_μ is a gauge field ($\mu = 0, 1, 2, 3$). Note that here $P_0 = i\hbar I_{n\times n}\otimes\partial/\partial q_0$. This is the basis for the so-called *gauge principle* whereby the form of the interaction is determined by local gauge invariance. The covariant derivative $D_\mu \equiv (i/\hbar)(P_\mu - gA_\mu)$ turns out to be of fundamental importance for determining the field strength tensor of the theory. It will be the operator that generalizes electromagnetic-like interactions.

To be specific let us consider a Dirac free particle described by the hamiltonian

$$H\Psi = (c\mathbf{\Sigma}\cdot\mathbf{P} + mc^2\beta)\Psi = i\hbar\frac{\partial\Psi}{\partial t}, \tag{6}$$

where

$$P_k \equiv \gamma_5 p_k = -i\hbar\gamma_5\nabla_k, \qquad \Sigma_k = \gamma_5\alpha_k = \begin{pmatrix}\sigma_k & 0\\ 0 & \sigma_k\end{pmatrix}, \tag{7}$$

with

$$\beta = \begin{pmatrix}I_{2\times2} & 0\\ 0 & -I_{2\times2}\end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix}0 & I_{2\times2}\\ I_{2\times2} & 0\end{pmatrix}. \tag{8}$$

Thus we define

$$\begin{aligned} Q_0 &\equiv I_{4\times4}\otimes q_0, \qquad Q_i \equiv -\gamma_5\otimes q_i,\\ P_0 &\equiv I_{4\times4}\otimes p_0, \qquad P_j \equiv -\gamma_5\otimes p_j, \end{aligned} \tag{9}$$

where $p_\mu = i\hbar\partial/\partial q^\mu$ and $\widehat{\eta} = -\gamma_5$. Thus $\Delta = 0$, i.e., n is an even integer and Q_i and P_j are formally "traceless" operators. The operators Q_μ and P_ν satisfy the canonical commutation relations

$$\begin{aligned} [Q_\mu, Q_\nu] &= [P_\mu, P_\nu] = 0,\\ [Q_\mu, P_\nu] &= i\hbar g_{\mu\nu}\mathbb{I}, \end{aligned} \tag{10}$$

where $\mathbb{I} = I_{4\times4}\otimes I$ and diag$(-1, 1, 1, 1)$ in the standard Dirac representation (Itzykson and Zuber, 1980). Here the normalization condition is

$$\int_{\mathbb{R}^3} \Psi^\dagger(\mathbf{q}, t)\Psi(\mathbf{q}, t)\, d^3q = 1. \tag{11}$$

A general interaction that keeps the $1/r$ dependence is introduced by means of the simultaneous (minimal) replacements

$$\text{(a)} \quad mc^2 \to mc^2 + \frac{\hbar c \varsigma}{q}, \tag{12}$$

$$\text{(b)} \quad P_0 \to P_0 - \frac{1}{c}\left(\frac{-Ze^2}{q}\right), \tag{13}$$

and

$$\text{(c)} \quad \mathbf{P} \to \mathbf{P} - i\frac{\lambda}{c}\beta\gamma_5\frac{\widehat{\mathbf{q}}}{q}, \tag{14}$$

where (a) is a *scalar* minimal interaction and (b) and (c) are *vector-like* minimal interactions. The wave equation for the interacting particle then becomes

$$\begin{aligned} H\Psi(\mathbf{q}, t) &= \left(c\boldsymbol{\Sigma} \cdot \left(\mathbf{P} - i\frac{\lambda}{c}\beta\gamma_5\frac{\widehat{\mathbf{q}}}{q}\right) + \beta\left(mc^2 + \frac{\hbar c\varsigma}{q}\right) - \frac{Ze^2}{q}\right)\Psi(\mathbf{q}, t) \\ &= i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{q}, t). \end{aligned} \tag{15}$$

Before solving the eigenvalue problem associated with (15), we recall that the operators

$$\widehat{K} \equiv \beta(\boldsymbol{\Sigma} \cdot \widehat{\mathbf{L}} + \hbar), \qquad \widehat{\mathbf{J}} \equiv \widehat{\mathbf{L}} + \frac{\hbar}{2}\boldsymbol{\Sigma}, \tag{16}$$

with $\widehat{\mathbf{L}} = \mathbf{q} \times \widehat{\mathbf{q}}$ the orbital angular momentum operator, are constants of motion: $[H, \widehat{K}] = \widehat{0}$, $[H, \widehat{\mathbf{J}}] = \widehat{0}$. Following a standard procedure (Itzykson and Zuber, 1980), the stationary states of energy E can be written as

$$\Psi_E(\mathbf{q}, t) = \begin{pmatrix} \psi_a(\mathbf{q}, t) \\ \psi_b(\mathbf{q}, t) \end{pmatrix} = \begin{pmatrix} \psi_a(q)\mathcal{Y}_{jj_3l_a}(\widehat{\mathbf{q}}) \\ i\psi_b(q)\mathcal{Y}_{jj_3l_b}(\widehat{\mathbf{q}}) \end{pmatrix} \exp\left(-\frac{i}{\hbar}Et\right), \tag{17}$$

where $\mathcal{Y}_{jj_3l}$ are the normalized total angular momentum functions, with

$$\widehat{\mathbf{L}}^2\mathcal{Y}_{jj_3l} = \hbar^2 l(l+1)\mathcal{Y}_{jj_3l}, \quad \widehat{\mathbf{J}}^2\mathcal{Y}_{jj_3l} = \hbar^2 j(j+1)\mathcal{Y}_{jj_3l}, \quad \widehat{K}\mathcal{Y}_{jj_3l} = \hbar\kappa\mathcal{Y}_{jj_3l}, \tag{18}$$

where $l_a = j \pm 1/2$ and $l_b = j \mp 1/2$ when $\kappa = \pm(j + 1/2)$. Thus the Dirac equation is equivalent to the set of first-order (nonlinear) differential equations

$$\begin{aligned} \left(\frac{d}{dq} + \frac{(1-\kappa)}{q}\right)f &= \left(\left(\frac{mc^2 - E}{\hbar c} + \frac{\varsigma}{q} - \frac{Z\alpha}{q}\right)g - \frac{\lambda}{q}f\right), \\ \left(\frac{d}{dq} + \frac{(1+\kappa)}{q}\right)g &= \left(\left(\frac{mc^2 + E}{\hbar c} + \frac{\varsigma}{q} + \frac{Z\alpha}{q}\right)f - \frac{\lambda}{q}g\right), \end{aligned} \tag{19}$$

where we have used the fact that

$$\hat{\mathbf{q}} \cdot \sigma \mathcal{Y}_{jj_3l_a}(\Omega) = -\mathcal{Y}_{jj_3l_b}(\Omega), \qquad \hat{\mathbf{q}} \cdot \sigma \mathcal{Y}_{jj_3l_b}(\Omega) = -\mathcal{Y}_{jj_3l_a}(\Omega). \tag{20}$$

By setting

$$f(q) \equiv \frac{F(q)}{q}, \qquad g(q) \equiv \frac{G(q)}{q}, \tag{21}$$

and

$$M_1 = \frac{mc^2 + E}{\hbar c}, \qquad M_2 = \frac{mc^2 - E}{\hbar c}, \tag{22}$$

$$r = \sqrt{M_1 M_2}|\mathbf{q}|, \quad Z\alpha = Z\frac{e^2}{\hbar c}, \quad b = \sqrt{M_1 M_2}, \tag{23}$$

we find that

$$\frac{dF}{dr} - \kappa\frac{F}{r} = \left(\sqrt{\frac{M_1}{M_2}} + \frac{\varsigma}{r} - \frac{Z\alpha}{r}\right)G - \frac{\lambda}{r}F$$

and (24)

$$\frac{dG}{dr} + \kappa\frac{G}{r} = \left(\sqrt{\frac{M_2}{M_1}} + \frac{\varsigma}{r} + \frac{Z\alpha}{r}\right)F - \frac{\lambda}{r}G.$$

Next we look for solutions in the form of series

$$G(r) = \exp(-r)r^s \sum_{\mu=0} b_\mu r^\mu, \qquad F(r) = \exp(-r)r^s \sum_{\mu=0} a_\mu r^\mu. \tag{25}$$

From (24) and (25) we obtain the recursion relation

$$(s + \mu - \kappa + \lambda)a_\mu + (Z\alpha - \varsigma)b_\mu - a_{\mu-1} = \sqrt{\frac{M_2}{M_1}}b_{\mu-1},$$

(26)

$$(s + \mu + \kappa + \lambda)b_\mu - (Z\alpha + \varsigma)a_\mu - b_{\mu-1} = \sqrt{\frac{M_1}{M_2}}a_{\mu-1}.$$

For $\mu = 0$,

$$((s - \kappa) + \lambda)a_0 + (Z\alpha - \varsigma)b_0 = 0$$

and (27)

$$((s + \kappa) + \lambda)b_0 - (Z\alpha - \varsigma)a_0 = 0,$$

i.e.,

$$s^2 + 2s\lambda + (Z\alpha)^2(\lambda + 1) - (\varsigma^2 + \kappa^2) = 0. \tag{28}$$

Given that $a_0, b_0 \neq 0$, from (28) we obtain

$$s = s_{\pm} = -\lambda \pm \sqrt{\varsigma^2 + \kappa^2 - Z^2\alpha^2} > -\frac{1}{2}. \tag{29}$$

Choosing $\mu = n' + 1$ and $a_{n'+1} = b_{n'+1} = 0$, to terminate the series, we have that $a_{n'} = -b_{n'}\sqrt{M_2/M_1}$. Then from (26) we get

$$2\sqrt{M_1 M_2}(s + n' + \lambda) = (M_1 - M_2)\, Z\alpha - (M_1 + M_2)\varsigma, \tag{30}$$

where

$$n \equiv n' + |\kappa| = n' + j + \frac{1}{2} \tag{31}$$

is the *principal quantum number*. By defining

$$\gamma = \frac{Z\alpha}{s + n - |\kappa| + \lambda}, \quad \xi = \frac{\varsigma}{s + n - |\kappa| + \lambda}, \quad \epsilon = \frac{E}{mc^2}, \tag{32}$$

we solve for ϵ:

$$1 - \epsilon^2 = (\epsilon\gamma - \xi)^2. \tag{33}$$

The solutions are

$$\epsilon_{\pm} = \frac{1}{1 + \gamma^2}\left(\gamma\xi \pm \sqrt{1 + \gamma^2 - \xi^2}\right) > 0.$$

Note that for the point nucleus there exist bound solutions for

$$1 + \gamma^2 - \xi^2 > 0, \tag{34}$$

or equivalently

$$(s + n - |\kappa| + \lambda)^2 > \varsigma^2 - (Z\alpha)^2. \tag{35}$$

Thus for ϵ_- we have the constraint

$$(\gamma\xi)^2 > 1 + \gamma^2 - \xi^2 > 0. \tag{36}$$

Explicitly we finally find the energy eigenvalues

$$E_{\pm} = \frac{mc^2(s + n - |\kappa| + \lambda)^2}{(s + n - |\kappa| + \lambda)^2 + (Z\alpha)^2} \times \left(\frac{Z\alpha\varsigma}{(s + n - |\kappa| + \lambda)^2} \pm \sqrt{1 + \left(\frac{(Z\alpha)^2 - \varsigma^2}{(s + n - |\kappa| + \lambda)^2} \right)} \right).$$

It is worth mentioning here that the Dirac oscillator (Moshinsky and Szczepaniak, 1989) can also be reobtained through a minimal interaction of the form (14) in the Dirac equation. This is done by choosing $A_k(q) = iq_k\beta\gamma_5 m\omega/e$, where ω is the frequency for the oscillator. This gauge field gives rise to a harmonic oscillator with a strong spin–orbit coupling that introduces, as in the previous case, an infinite degeneracy. This oscillator has a hidden supersymmetry, responsible for the special properties of its spectrum (Benitez *et al.*, 1990). It is interesting to note that the vector field A_k in (14) is a Hermitian operator. This feature is absent in Moshinski's approach (Moshinsky and Szczepaniak, 1989).

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